Filtering With Confidence: In-sample Confidence Bands For GARCH Filters

Marcin Zamojski

University of Gothenburg

ES NASM, St. Louis
June 17, 2017

Paper/slides @ **zamojski.net**

UNIVERSITY OF GOTHENBURG
SCHOOL OF BUSINESS, ECONOMICS AND LAW

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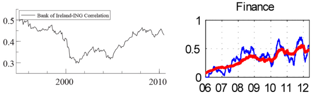
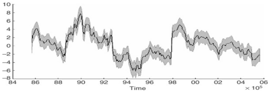


Term structure of interest rates
slope from Diebold, Li, Yue (2008 JE)



Systemic risk
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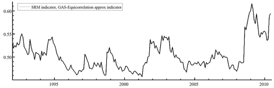
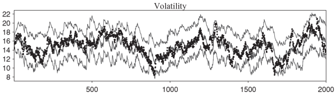
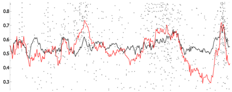


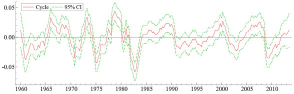
Stochastic Volatility with jumps
Johannes, Polson, Stroud (2009 RFS)



Loss-Given-Default
Creal, Schwaab, Koopman, and Lucas (2014 REStat)



Business and financial cycles
Galati, Hindrayanto, Koopman, and Vlekke (2015 EL)



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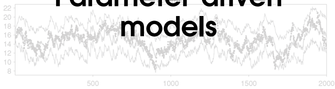

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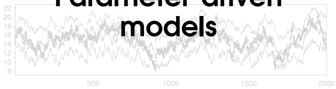

Q: Is it possible to plot confidence bands around such paths?

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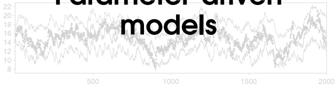

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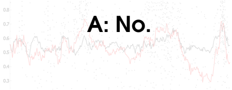

A: No.

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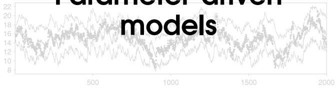

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Motivation, main results

GARCH(1,1) volatility of S&P 500 returns. 2008–2012.

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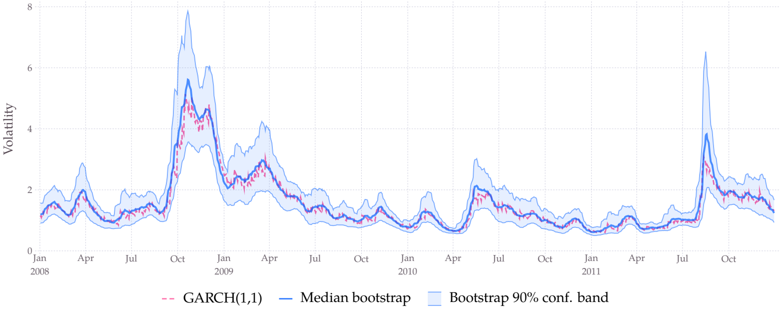

Filtering With Confidence

- A novel bootstrap + a new resampling scheme.

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Filtering With Confidence

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- Volatility modelling
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- No time ☹
- Conclusions

Observation- vs. parameter-driven methods

Observation-driven

Observation equation:

$$y_t \sim p\left(y_t | Y_{t-1}, \boldsymbol{f}_t; \boldsymbol{\theta}\right),$$

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The usual treatment imposes a linear, autoregressive updating function with an innovation equal to the score of the criterion function with respect to the time-varying parameter, $\boldsymbol{f}_t$.

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The gist: a Gauss-Newton update in the time dimension based on the locally observed misfit.

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GARCH(1,1)

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\varepsilon_t &= \eta_t \sqrt{\sigma_t^2}, \qquad \eta_t \sim \mathrm{N}(0,1)\,, \\
\sigma_{t+1}^2 &= \omega + \beta\sigma_t^2 + \alpha y_t^2, \\
\sigma_{t+1}^2 &= \omega + (\alpha+\beta)\sigma_t^2 + \alpha\left(y_t^2 - \sigma_t^2\right).
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Stochastic volatility

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Nelson (1992, JE) shows that GARCH(1,1) produces consistent paths, even if the true DGP is the SV.

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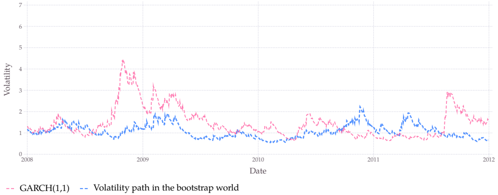

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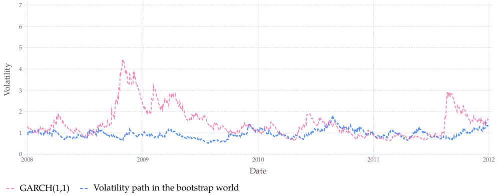

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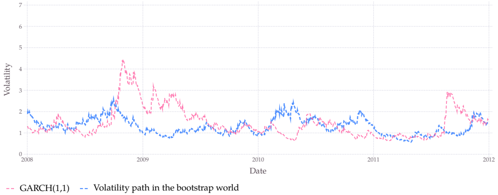

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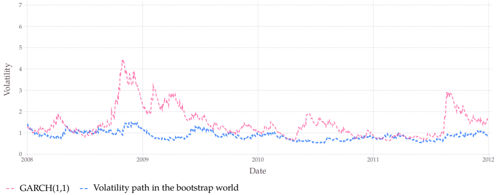

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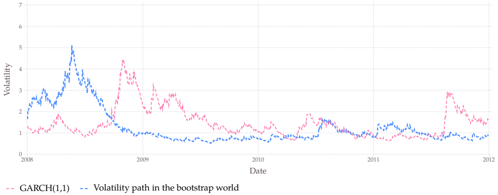

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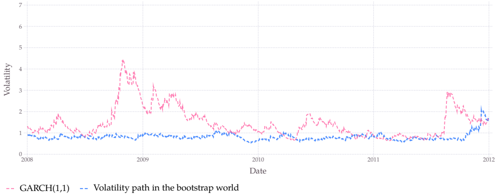

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$$y_t = \varepsilon_t = \eta_t \sqrt{\sigma_t^2}, \qquad \eta_t \sim \mathrm{N}(0, 1), \qquad \sigma_{t+1}^2 = \omega + \beta \sigma_t^2 + \alpha y_t^2$$

1. Use GARCH(1,1) to get $\hat{\sigma}_t^2$ and fitted residuals $\hat{\varepsilon}_t$, get raw residuals: $\hat{\eta}_t = \hat{\varepsilon}_t \left(\hat{\sigma}_t\right)^{-1}$.
2. Resample raw residuals treating them as *i.i.d.* to get *B* bootstrap samples.
3. Use the GARCH(1,1) transition equation to get new paths of volatility, ${\sigma_{t+1}^2}^* = \hat{\omega} + \hat{\beta} \sigma_t^{2*} + \hat{\alpha} y_t^{2*}$.
4. Fit GARCH(1,1) to all bootstrap samples.
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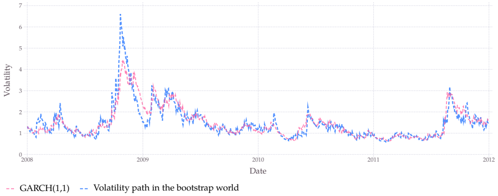

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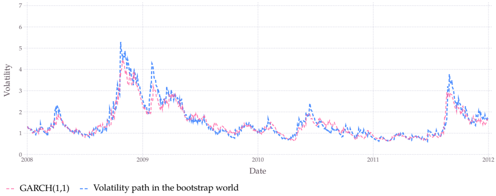

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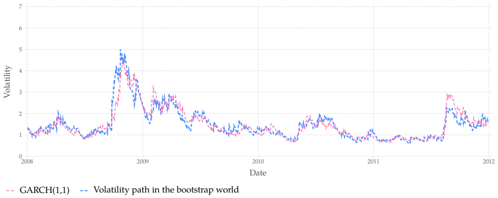

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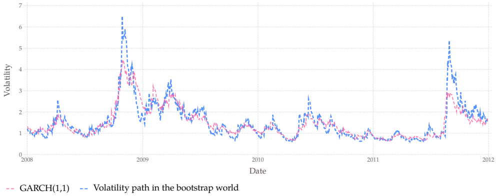

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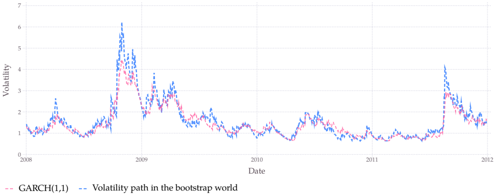

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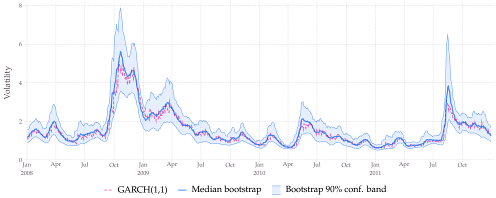

LITE resampler

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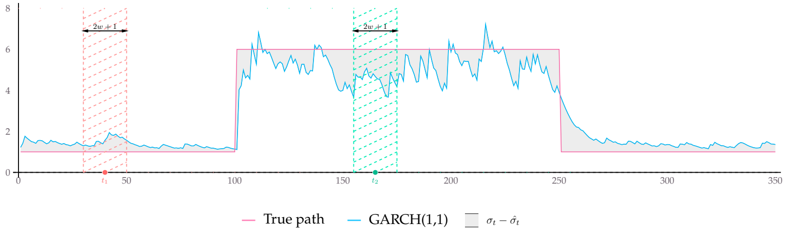
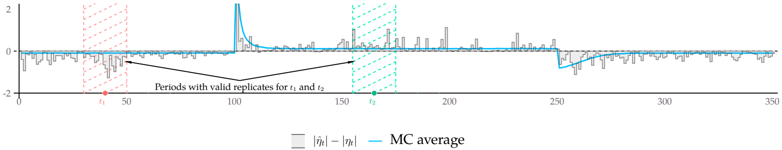
Instead of *i.i.d.*, the LITE resamples residuals in a moving-window fashion. LITE $\equiv$ Local In TimE.



In the neighbourhood of t_1 and t_2 fit of the model is considerably different.



Optimal bandwidth ('best' coverage)

Minimize average bias of bootstrap paths w.r.t. the GARCH path:

$$\underset{b:\delta(t)\sim\Delta_t^b}{\text{argmin}} \left\{ \frac{1}{n} \sum_t \left[\frac{1}{B} \sum_{i=1}^{B} \hat{\sigma}_t^{2\,*} - \hat{\sigma}_t^2 \right]^2 \right\}.$$

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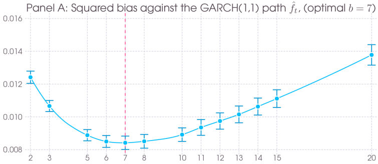

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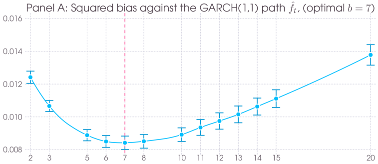

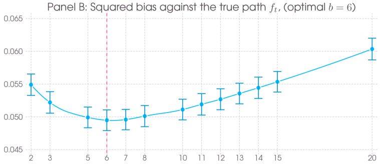

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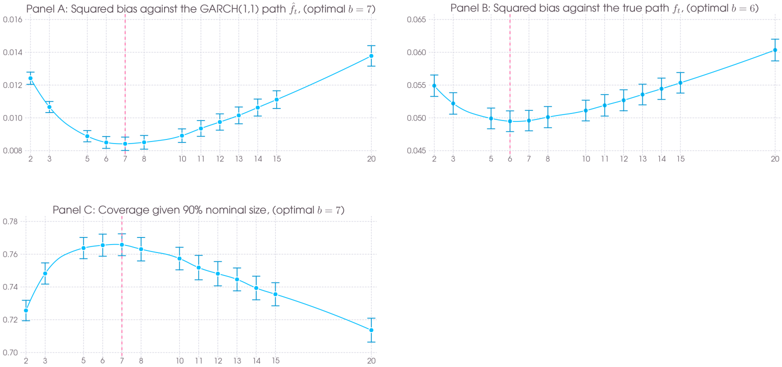

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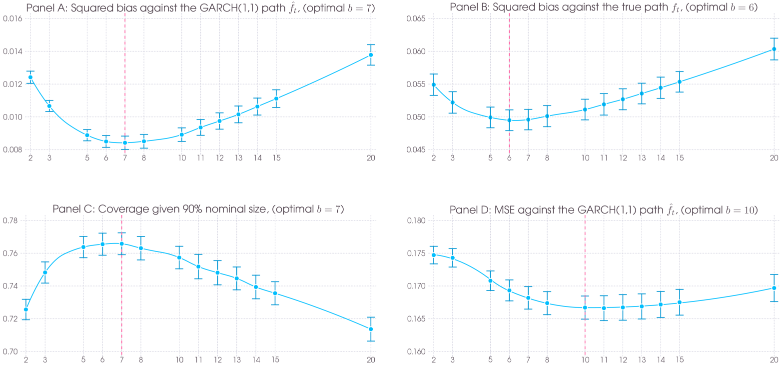


Empirical application

- S&P 500 returns.
- Period: 1980–2015 (9040 daily observations).
- Total return index from Datastream.
- GARCH(1,1) parameters are (with values of the *t*-statistic in parentheses):

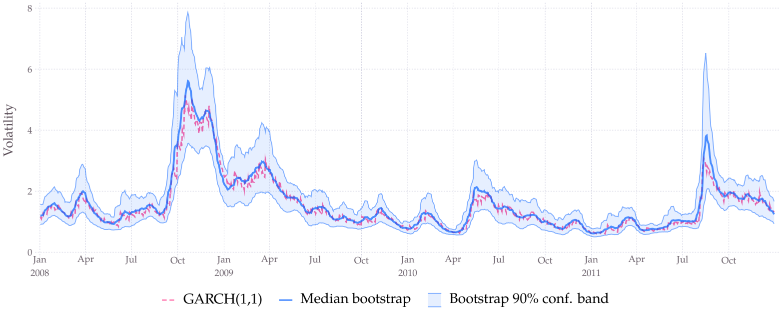
$$\omega = 0.0145(6.5652),$$

$$\alpha = 0.0789(13.626),$$

$$\beta = 0.910(128.6).$$

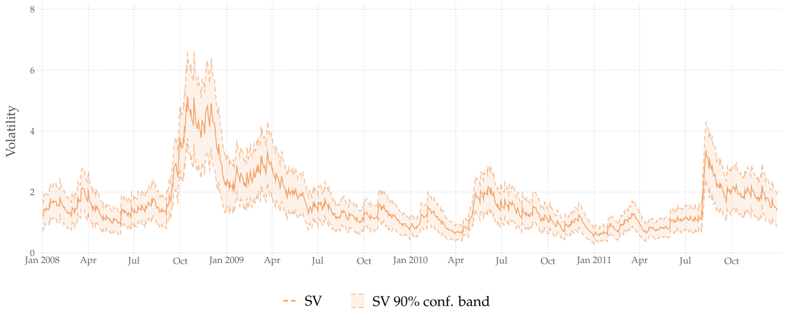
Modelling volatility, empirical results

GARCH(1,1)



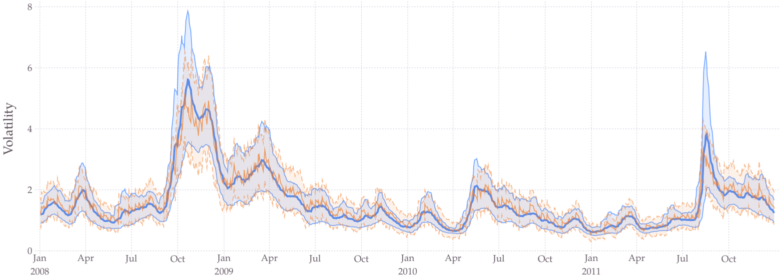
Modelling volatility, empirical results (benchmarked)

Stochastic volatility



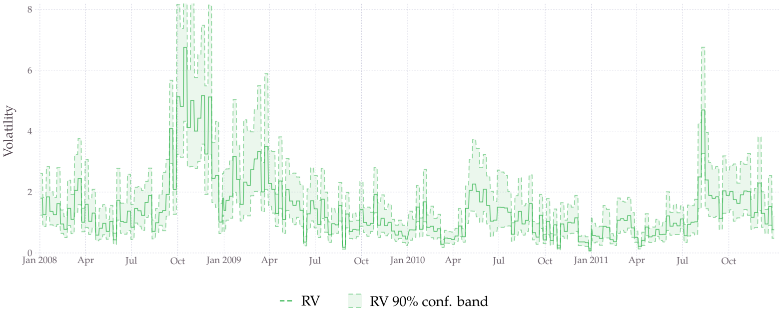
Modelling volatility, empirical results (benchmarked)

GARCH(1,1) vs. Stochastic volatility



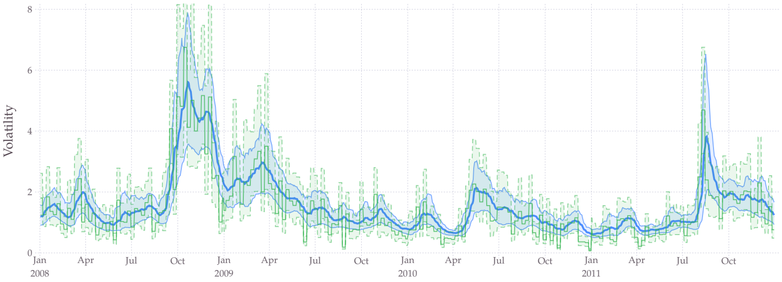
Modelling volatility, empirical results (benchmarked)

Realized volatility

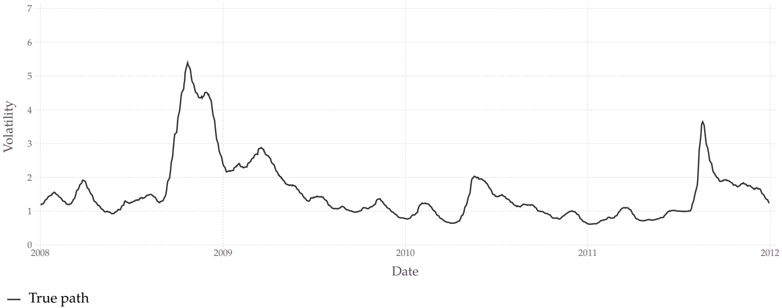


Modelling volatility, empirical results (benchmarked)

GARCH(1,1) vs. Realized volatility



LITE bootstrap + LITE resampler, simulations



A single path of volatility (deterministic or from an SV model) shared across 1,000 Monte Carlo replications. Innovations in the observation equation are drawn as *i.i.d.* N(0,1).

$$y_t = \varepsilon_t,$$

$$\varepsilon_t = \eta_t \sigma_t, \qquad \eta_t \sim \mathrm{N}(0, 1).$$

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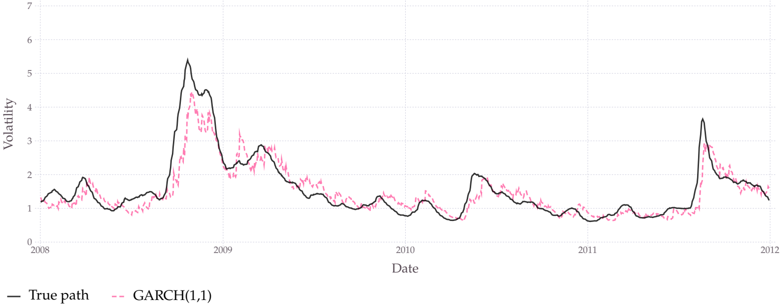

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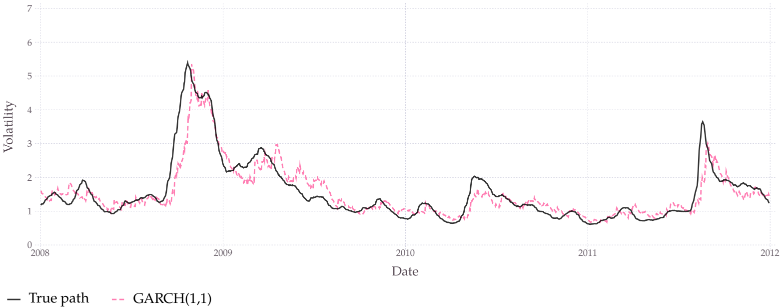

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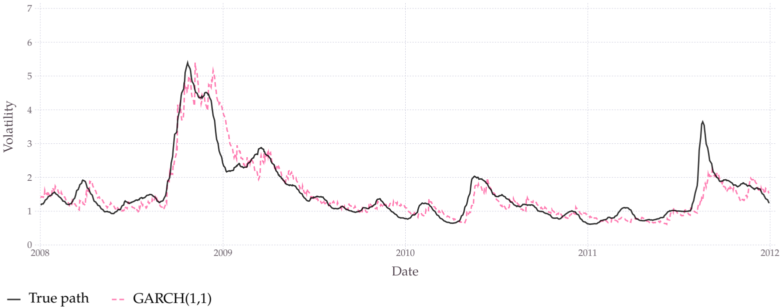

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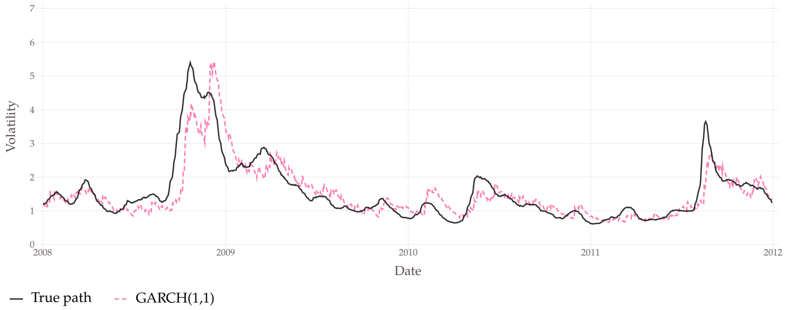

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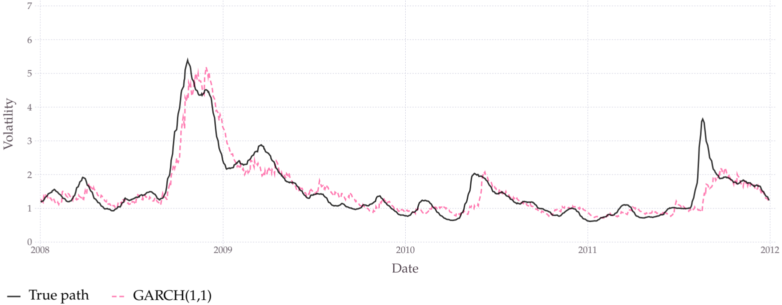

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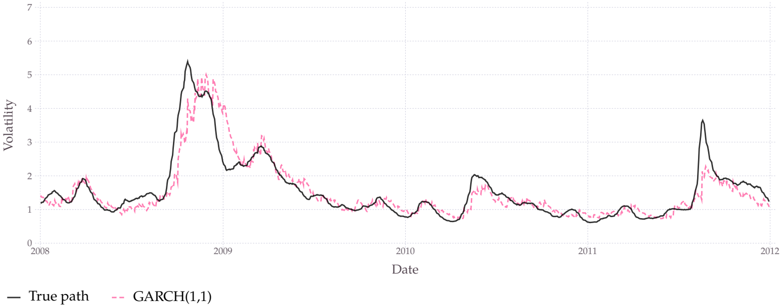

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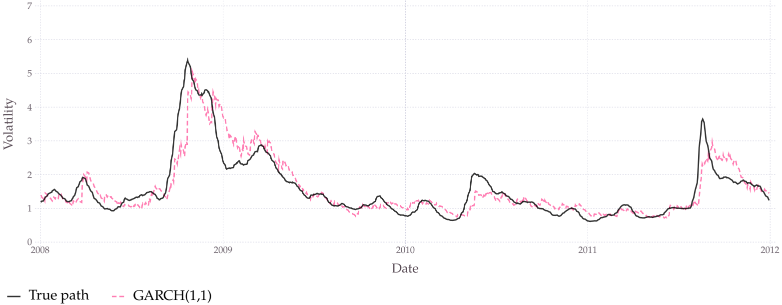

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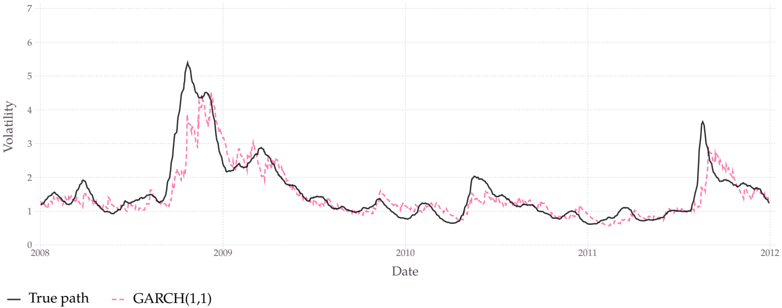

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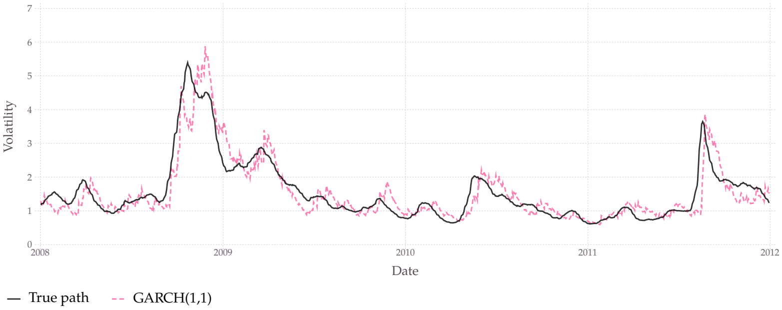

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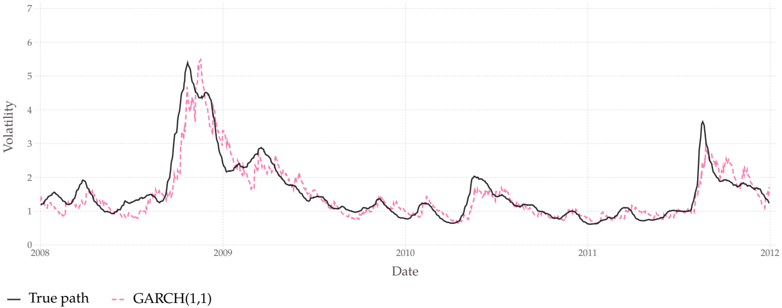

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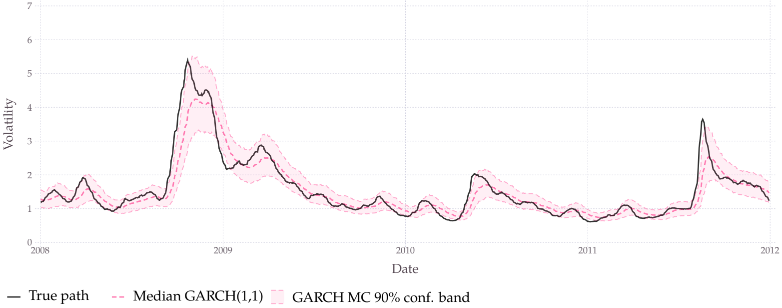
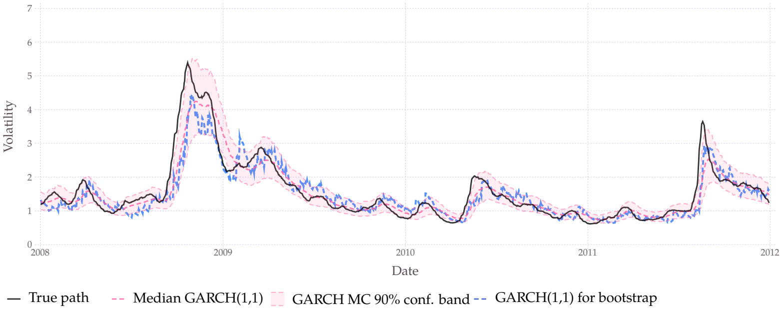


Given the 1,000 GARCH paths, I construct a 'GARCH MC' confidence band as a benchmark. This is the best the GARCH filter can be expected to do.

LITE bootstrap + LITE resampler, simulations



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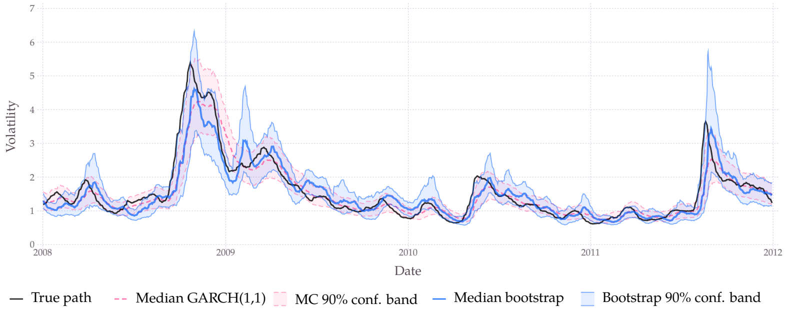
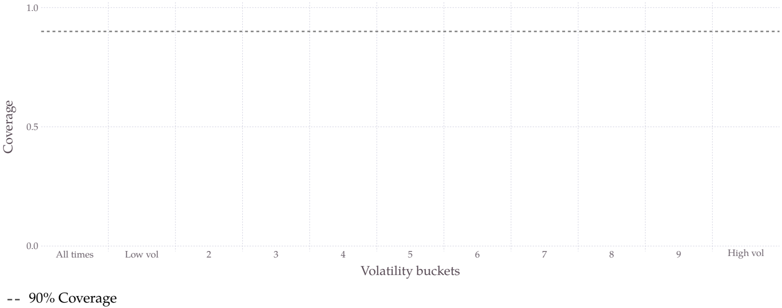
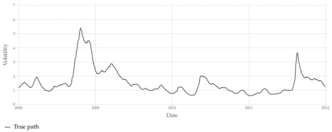


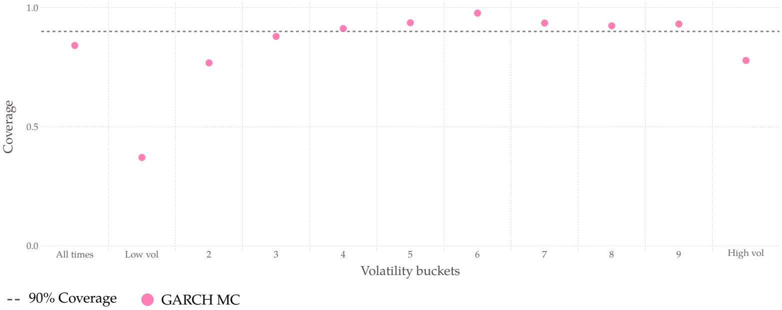
LITE bootstrap + LITE resampler, coverage

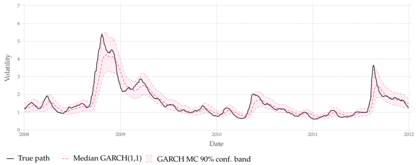


Average path coverage: $C = \frac{\#(\text{in the band})}{T}$.



LITE bootstrap + LITE resampler, coverage





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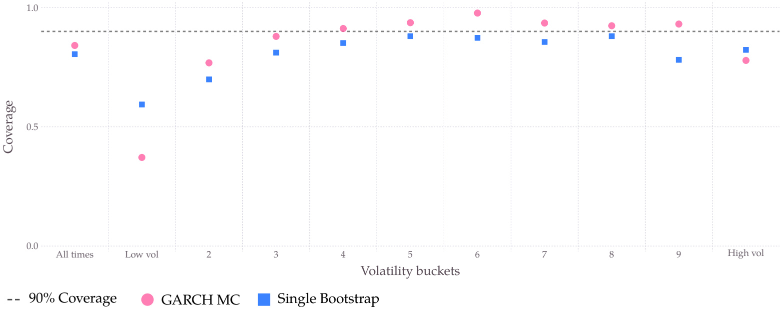

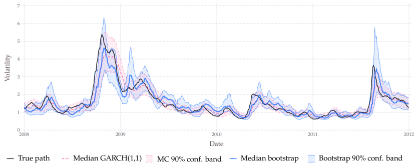

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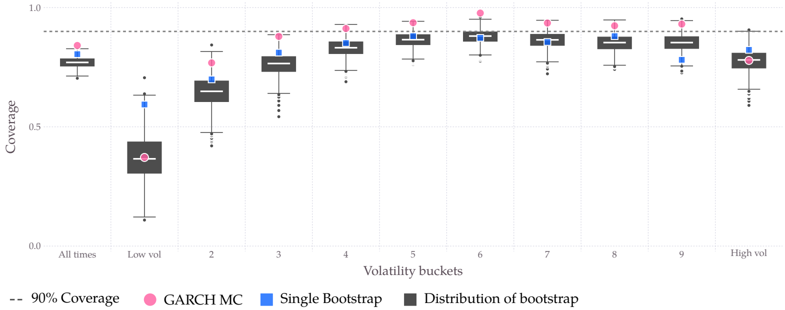
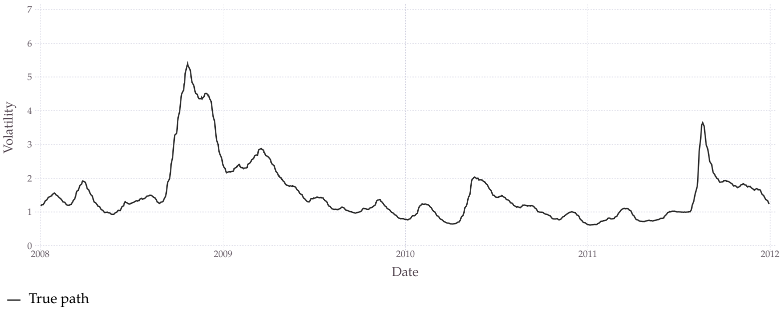
LITE bootstrap + LITE resampler, coverage



- LITE bands have unconditional coverage of ca. 77%.
- The benchmark, GARCH MC, is at 84%.
- Coverage is not uniform and depends on the value of volatility itself.

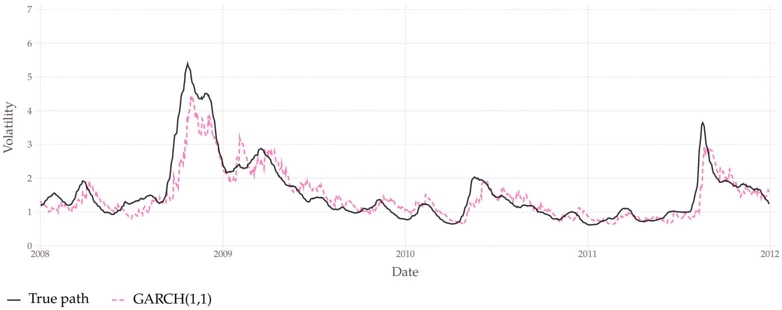
LITE bootstrap + LITE resampler, convergence



Increase number of observations from 6,952 days to 50,000.

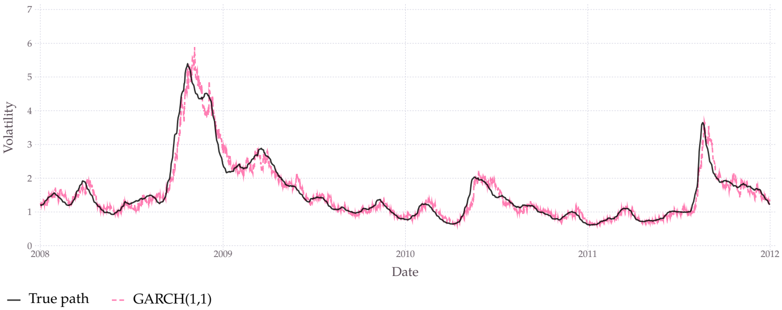
LITE bootstrap + LITE resampler, convergence

Daily, 6,952 observations



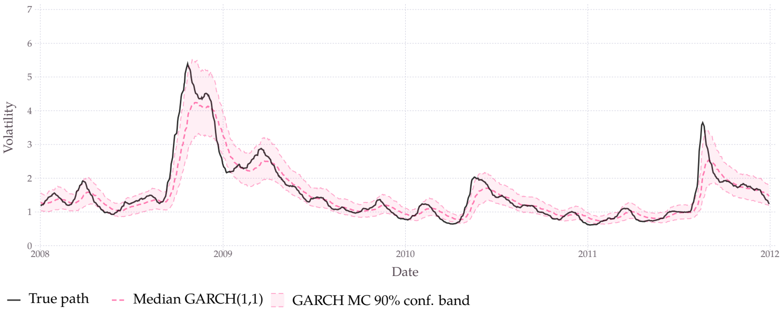
LITE bootstrap + LITE resampler, convergence

7x, 50,000 observations



LITE bootstrap + LITE resampler, convergence

Daily, 6,952 observations



LITE bootstrap + LITE resampler, convergence

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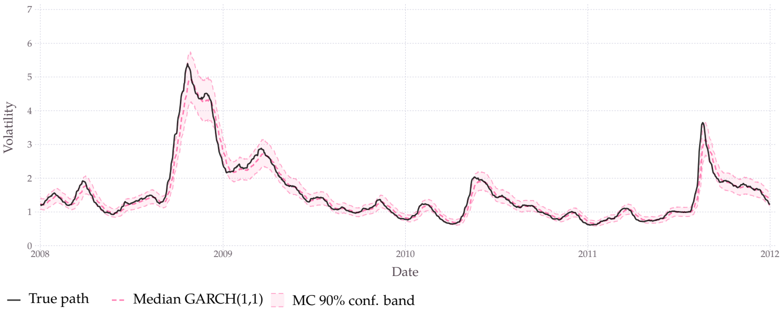

LITE bootstrap + LITE resampler, convergence

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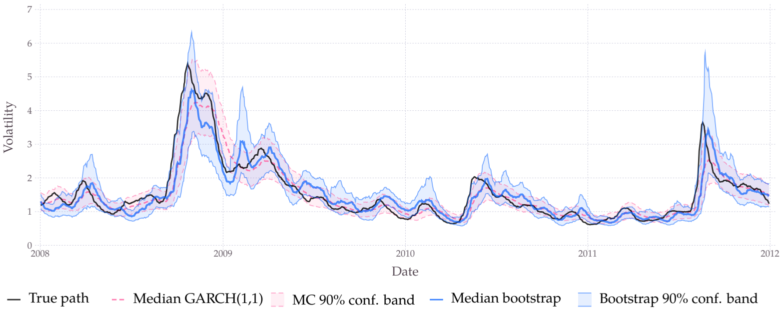

LITE bootstrap + LITE resampler, convergence

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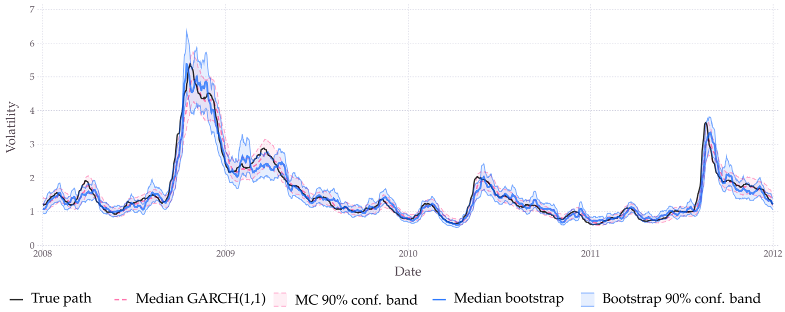

LITE bootstrap + LITE resampler, convergence

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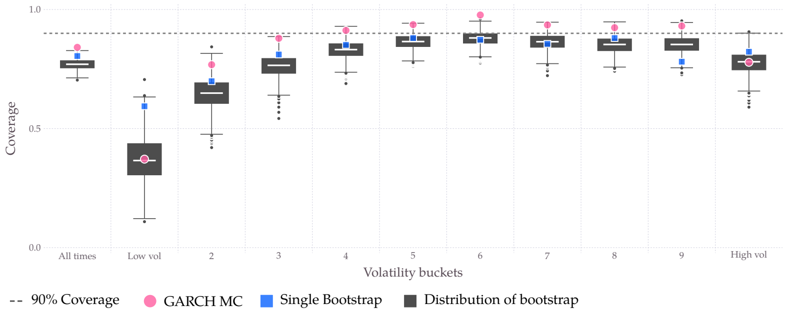

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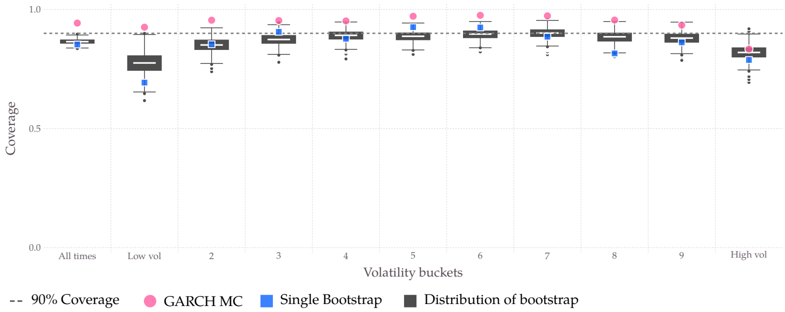
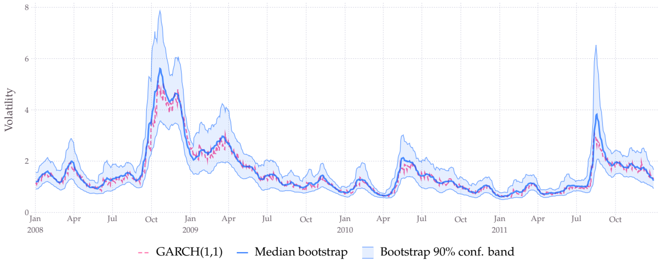


Applications?

- Risk budgeting
- Ensemble estimators
- VaR, ES
- Trading (based on span or asymmetry)



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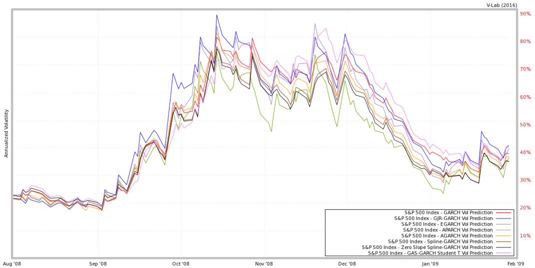

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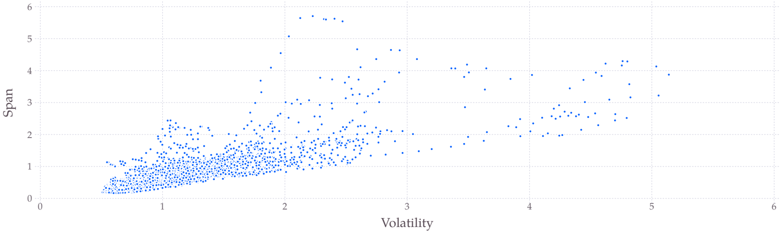
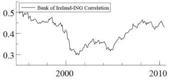
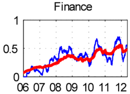


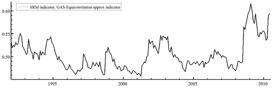
Applications?

Correlation
Lucas, Schwaab, and Zhang (2015 JAE), Oh and Patton (2013)

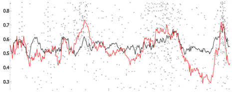




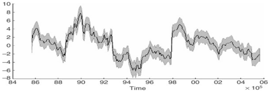
Systemic risk
Lucas, Schwaab, and Zhang (2015 JAE), Oh and Patton (2013)



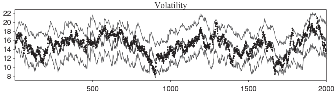
Loss-Given-Default
Creal, Schwaab, Koopman, and Lucas (2014 REStat)



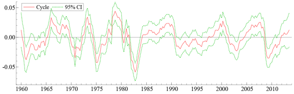
Term structure of interest rates
slope from Diebold, Li, Yue (2008 JE)



Stochastic Volatility with jumps
Johannes, Polson, Stroud (2009 RFS)



Business and financial cycles
Galati, Hindrayanto, Koopman, and Vlekke (2015)



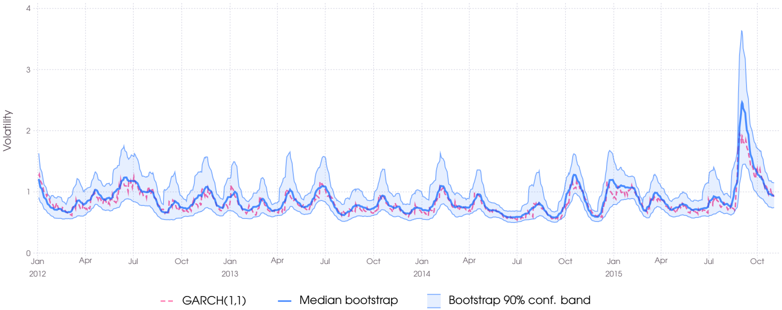
Conclusions

- A novel bootstrap approach combined with a new resampling scheme.
- Produces in-sample confidence bands around time-varying parameters in observation-driven models.
- Accounts for parameter and filtering uncertainty.
- Average coverage converges to the nominal level.
- Easily implementable and a flexible method.
- Can be used as a smoother to mitigate attenuation bias.

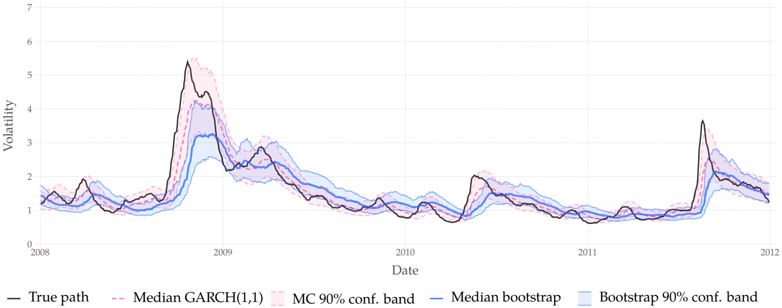
Really easy

```
function sample_locally(rng, b)
    out=collect(rng)
    out[b+1:N-b]=sample(-b:b, length(b+1:N-b))+out[b+1:N-b]
    for i in unique([1:b; N-b+1:N])
        if i>=1 && i<=N
            rng=maximum([1,i-b]):minimum([N,i+b])
            out[i]=sample(rng,1)[1]
        end
    end
    return out
end

σ, ε, params=fit_garch(params, y);

#Normalized residuals
normalized_residuals= ε ./ σ;
normalized_residuals-=mean(normalized_residuals);

#Running bootstrap
paths=zeros(T,B+1); paths[:, end]=σ;
for i=1:B
    yᴮ=normalized_residuals[sample_locally(1:T, bandwidth)] .* σ;
    σᴮ, εᴮ, paramsᴮ=fit_garch(params, yᴮ);
    paths[i,:]=σᴮ;
end
get_bands(paths)
```

S&P 500 in 2012–2015



LITE bootstrap + *I.I.D.* resampler $\rightarrow$ mess



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